

THE SPACE OF LINEAR MAPS INTO A GRASSMANN MANIFOLD

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ABSTRACT. We show that the space of all holomorphic maps of degree one from the Riemann sphere into a Grassmann manifold is a sphere bundle over a flag manifold. Using the notions of “kernel” and “span” of a map, we completely identify the space of unparameterized maps as well. The illustrative case of maps into the quadric Grassmann manifold is discussed in details and the homology of the corresponding spaces computed.

1. INTRODUCTION

Given a complex projective variety M , one can consider the space $\text{Hol}(M)$ of all holomorphic maps from the Riemann sphere \mathbb{P}^1 into it. This can be given the structure of a quasiprojective variety [11] and thus has the homotopy type of a finite CW complex. If we fix basepoints in all of \mathbb{P}^1 and M , then we denote by $\text{Rat}(M)$ the subspace of basepoint preserving maps. The homeomorphism type of this subspace doesn't depend on the choice of basepoint in \mathbb{P}^1 and if M is homogeneous, it is independent of the choice of basepoint in M as well.

The space $\text{Hol}(M)$ is not in general connected. Fixing $\alpha \in H_2(M)$, one defines $\text{Hol}_\alpha(M)$ to be the subspace of all morphisms $f : \mathbb{P}^1 \longrightarrow M$ with $f_*([\mathbb{P}^1]) = \alpha$. In [21] and for $M = G/P$, P a parabolic subgroup, sufficient conditions on α are given so that $\text{Hol}_\alpha(G/P)$ is irreducible and smooth.

When $M = \text{Gr}(n, m)$ is the Grassmann manifold of n -dimensional complex linear planes in \mathbb{C}^{n+m} , we say that a map $f : \mathbb{P}^1 \longrightarrow \text{Gr}(n, m)$ has degree $d \geq 0$ if its effect on the second homology group is multiplication by d . It turns out that two maps in $\text{Hol}(\text{Gr}(n, m))$ are in the same connected component if and only if they have the same degree. We write $\text{Hol}_d(\text{Gr}(n, m))$ the component of degree d maps. Of course $\text{Hol}_0(\text{Gr}(n, m)) = \text{Gr}(n, m)$ are the constant maps, while $\text{Hol}_1(\text{Gr}(n, m))$ are the linear maps which we study in details in this paper. The following is a complete description of this space.

Theorem 1.1. *Let $Fl_{(1,n)}(\mathbb{C}^{n+m}) = U(n+m)/U(1) \times U(n-1) \times U(m)$ be the flag manifold of all $(1, n)$ flags in \mathbb{C}^{n+m} with projections*

$$\begin{aligned} p_1 & : Fl_{(1,n)}(\mathbb{C}^{n+m}) \longrightarrow \mathbb{P}^{n+m-1} \\ p_2 & : Fl_{(1,n)}(\mathbb{C}^{n+m}) \longrightarrow \text{Gr}(n, m) \end{aligned}$$

Then $\text{Hol}_1(\text{Gr}(n, m))$ has as deformation retract the total space of the sphere bundle over $Fl_{(1,n)}(\mathbb{C}^{n+m})$ associated to the rank m complex vector bundle

$$p_1^*(H) \otimes p_2^*(Q)$$

where $H = \mathcal{O}(1)$ is the line bundle dual to the tautological line bundle over \mathbb{P}^{n+m-1} , and Q the anti-tautological bundle over $\text{Gr}(n, m)$.

Some descriptions of $\text{Hol}(\text{Gr}(n, m))$ as an algebraic variety can be found in [11, 12, 16] but no homotopy type has been computed. Note that the space $\text{Rat}(\text{Gr}(n, m))$ plays a significant role in the theory of multilinear control systems. It corresponds to the moduli space of n -input, m -output time invariant systems that are controllable and observable [14] (and references therein). A complete and elegant computation of the homology of $\text{Rat}_d(\text{Gr}(n, m))$ for all positive d was obtained by Mann and Milgram [19, 20]. The full space $\text{Hol}_d(\text{Gr}(n, m))$ corresponds on the other hand to “singular” or “generalized” state space systems [6]. A more detailed study of $\text{Hol}_d(\text{Gr}(n, m))$ for $d > 1$ will be pursued elsewhere.

Observe that $\text{Hol}_1(\mathbb{P}^1) = \text{PGL}_2(\mathbb{C}) \simeq \text{PU}(2)$ acts freely on $\text{Hol}_1(\text{Gr}(n, m))$ by precomposition of maps. Similarly in the based case, there is an action of $\text{Rat}_1(\mathbb{P}^1) = \text{Aff}(\mathbb{C})$ on $\text{Rat}_1(\text{Gr}(n, m))$. Here $\text{Aff}(\mathbb{C}) = \{z \mapsto az + b, a \in \mathbb{C}^*, b \in \mathbb{C}\}$ is a semi-direct product and $\text{Aff}(\mathbb{C}) \simeq S^1$. Define $\widetilde{\text{Rat}}_1(\text{Gr}(n, m)) := \text{Rat}_1(\text{Gr}(n, m))/\text{Aff}(\mathbb{C})$ to be the space of *unparameterized* maps. Similarly define $\widetilde{\text{Hol}}_1(\text{Gr}(n, m)) := \text{Hol}_1(\text{Gr}(n, m))/\text{PGL}_2(\mathbb{C})$. Our next result determines completely the homeomorphism type of these spaces and makes the cute observation that an element in $\widetilde{\text{Hol}}_1(\text{Gr}(n, m))$ is completely determined by its “span” and its “kernel”. Following ([9], p. 526), we define the “kernel” and “span” of a holomorphic map $f : \mathbb{P}^1 \longrightarrow \text{Gr}(n, m)$ to be respectively the intersection and span of all vector subspaces $f(p) \subset \mathbb{C}^{n+m}$, $p \in \mathbb{P}^1$ (see §5).

Theorem 1.2. *By sending a holomorphic map f to the flag $(\ker(f) \subset \text{span}(f) \subset \mathbb{C}^{n+m})$, we construct homeomorphisms*

$$\begin{aligned} \widetilde{\text{Hol}}_1(\text{Gr}(n, m)) &\xrightarrow{\cong} \text{Fl}_{(n-1, n+1)}(\mathbb{C}^{n+m}) \\ \widetilde{\text{Rat}}_1(\text{Gr}(n, m)) &\xrightarrow{\cong} \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \end{aligned}$$

where $\text{Fl}_{(n-1, n+1)}(\mathbb{C}^{n+m})$ is the variety of all $(n-1, n+1)$ -flags in \mathbb{C}^{n+m} of complex dimension $nm + n + m - 3$.

A decent part of this paper studies the special case of holomorphic maps into the quadric grassmann manifold $\text{Gr}(2, 2)$. This is the variety realized via the Plücker embedding as a quadric hypersurface in \mathbb{P}^5 . It has the homotopy and homology groups of $\mathbb{P}^2 \times S^4$ (cf. §6.1, [18]). We shall prove that

Theorem 1.3.

- (i) *There is a homotopy equivalence $\text{Rat}_1(\text{Gr}(2, 2)) \simeq S^2 \times S^3$ (Corollary 2.4)*
- (ii) *There is a (non-multiplicative) homotopy equivalence (Proposition 3.3)*

$$\Omega(\text{Gr}(2, 2)) \simeq S^1 \times S^3 \times \Omega S^5 \times \Omega S^7$$

- (iii) *The cohomology groups $\tilde{H}^*(\text{Hol}_1(\text{Gr}(2, 2)); \mathbb{Z})$ are given by (Theorem 6.9)*

i	2	4	6	7	8	9	11	13
H^i	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}_4	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}

Part (iii) follows from a careful analysis of the Gysin sequence associated to the bundle given in Theorem 1.1 and from the classical description of P. Baum of the cohomology of flag varieties. This allows for effective homology computations and for a complete description of the differentials in the evaluation fibration $\text{Rat}_1(\text{Gr}(2, 2)) \longrightarrow \text{Hol}_1(\text{Gr}(2, 2)) \xrightarrow{ev} \text{Gr}(2, 2)$ which would have been otherwise quite difficult to obtain [1].

Remark 1.4. Write ι the inclusion of based holomorphic maps into all based continuous maps

$$\iota : \text{Rat}_1(\text{Gr}(n, m)) \hookrightarrow \Omega_1^2(\text{Gr}(n, m)) \quad , \quad 1 \leq n \leq m$$

The induced map in homology is determined in [20]. This is a non-trivial calculation and aspects of this are discussed in §3. The important claim made in [20] is that the homology of $\text{Rat}_k(\text{Gr}(n, m))$ is generated via homology operations by the homology of $\text{Rat}_1(\text{Gr}(n, m))$. This point we shall return to in the future.

2. THE BASED LINEAR MAPS

The Grassmann manifold $\text{Gr}(n, m)$ is the homogeneous space $U(n+m)/U(n) \times U(m)$. It is a smooth complex variety of dimension nm . A system of charts for $\text{Gr}(n, m)$ is given as follows. Choose a decomposition of \mathbb{C}^{n+m} into a direct sum $\mathbb{C}^n \oplus \mathbb{C}^m$, and let $L : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be any linear operator. Then its graph is an n -dimensional subspace of \mathbb{C}^{n+m} . The set $\text{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ of all subspaces obtained in this way is an open dense subspace V of $\text{Gr}(n, m)$. Thus any point of V can be represented by an $m \times n$ complex matrix and this is a cell of dimension nm . By choosing various coordinate decompositions of \mathbb{C}^{n+m} , we can cover $\text{Gr}(n, m)$ with affine charts this way.

Identify $Gr(n, m)$ with all n -planes W in some decomposition $U \oplus Y$ where $U \cong \mathbb{C}^n$ and $Y \cong \mathbb{C}^m$. Then the maximal Schubert cell in $Gr(n, m)$ is identified with $Hom(U, Y)$ and its complement is a divisor (a Schubert hypersurface)

$$\Delta = \{W \mid \dim(W \cap Y) \geq 1\}$$

A based holomorphic map $f : \mathbb{P}^1 = \mathbb{C} \cup \infty \rightarrow Gr(n, m)$ can then be thought of as a *meromorphic* map into this Schubert cell $\mathbb{C} \rightarrow Hom(\mathbb{C}^n, \mathbb{C}^m)$, sending ∞ to the trivial matrix. The degree of such a map is computed as the intersection number with the Schubert hypersurface; $\deg f = (f(\mathbb{P}^1) : W)$. Such a map can be written as a *rational matrix*¹

$$(1) \quad f(z) = \sum \frac{A_i}{(z - z_i)^{k_i}}$$

with A_i an $m \times n$ matrix. The degree of the map in (1) depends on the multiplicity of the poles z_i and the ranks of the A_i [7].

Remark 2.1. There is another useful expression for f given in matrix form with entries in $\mathbb{C}[z]$. Indeed a holomorphic map $f : \mathbb{P}^1 \rightarrow Gr(n, m)$ sending ∞ to E_n ; the plane spanned by the *first* n -coordinates vectors, can be represented in the form

$$z \mapsto \text{span of row vectors of } [D : N]$$

where $[D : N]$ is an $n \times (n + m)$ matrix, uniquely defined up to $GL_n(\mathbb{C}[z])$ -multiplication on the left, with $D \in M_{n \times n}(\mathbb{C}[z])$, $N \in M_{n \times m}(\mathbb{C}[z])$ and the degree of the determinant of D is maximal among all $n \times n$ minor determinants. Out of this representation we can recover the transfer function described in (1) according to the formula $T(z) = D^{-1}N$. This correspondence is discussed in various books in linear control theory (see also [1, 19]). We will use both representations (as a matrix form or a transfer function) in §5.

Based on the transfer map description of based holomorphic maps (1), we give a shorter proof of the following observation due to Mann and Milgram.

Proposition 2.2. *$Rat_1(Gr(n, m))$ is homeomorphic to*

$$\mathbb{C} \times (\mathbb{C}^m - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^n - \{0\})$$

with \mathbb{C}^ acting diagonally; i.e. $a(\mathbf{v}, \mathbf{w}) = (a^{-1}\mathbf{v}, a\mathbf{w})$. This is up to homotopy the space of rank one matrices of size $m \times n$.*

Proof. Elements in $Rat_1(Gr(n, m))$ are maps of the form $z \mapsto \frac{A}{z - z_0}$, where A is a rank one $m \times n$ matrix. Any such matrix can be written as a product $\mathbf{v} \cdot \mathbf{w}^T$ where \mathbf{w}^T is a non-zero row vector of \mathbb{C}^n and \mathbf{v} is a non-zero column vector of \mathbb{C}^m . Note that $(a^{-1}\mathbf{v}, a\mathbf{w})$ with $a \in \mathbb{C}^*$ gives the same matrix $a^{-1}\mathbf{v} \cdot a\mathbf{w}^T = \mathbf{v} \cdot \mathbf{w}^T$. It is then immediate to see that the assignment

$$\begin{aligned} Rat_1(Gr(n, m)) &\xrightarrow{\cong} \mathbb{C} \times ((\mathbb{C}^m - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^n - \{0\})) \\ \frac{A}{z - z_0} &\mapsto (z_0, \mathbf{v} \times_{\mathbb{C}^*} \mathbf{w}) \end{aligned}$$

is a homeomorphism, where $A = \mathbf{v} \cdot \mathbf{w}^T$ is a rank one $m \times n$ matrix. □

Proposition (2.2) makes $Rat_1(Gr(n, m))$ into a bundle over \mathbb{P}^{n-1} with fiber $\mathbb{C}^m - \{0\} \simeq S^{2m-1}$. We can identify this fiber in a different way. Consider the inclusion $\iota_m : Rat_1(\mathbb{P}^m) \hookrightarrow Rat_1(Gr(n, m))$ which is induced from the inclusion $\mathbb{P}^m = Gr(1, m) \hookrightarrow Gr(n, m)$. According to the above Proposition, $Rat_1(\mathbb{P}^m) \simeq S^{2m-1}$ and in fact ι_m is up to homotopy the inclusion of a fiber.

Lemma 2.3. *There is a bundle*

$$Rat_1(\mathbb{P}^m) \xrightarrow{\iota_m} Rat_1(Gr(n, m)) \longrightarrow \mathbb{P}^{n-1}$$

¹Known in control theory as the “transfer matrix”

Proof. Using the affine coordinate description of $Gr(n, m)$ discussed above, the inclusion $\mathbb{P}^m \hookrightarrow Gr(n, m)$ is described at the level of a chart by the map $Hom(\mathbb{C}^1, \mathbb{C}^m) \longrightarrow Hom(\mathbb{C}^n, \mathbb{C}^m)$ which to $f : z \mapsto f(z)$ associates $(z_1, \dots, z_n) \mapsto f(z_1)$. At the level of matrices, if $f \in Rat_1(\mathbb{P}^m)$ is given by $z \mapsto \frac{1}{z-a}[a_1, \dots, a_m]^T$, then

$$\iota_m(f) : z \mapsto \frac{1}{z-a} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & & & \\ a_m & 0 & \cdots & 0 \end{pmatrix}$$

But if we identify $Rat_1(Gr(n, m))$ with $\mathbb{C} \times (\mathbb{C}^m - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^n - \{0\})$ as in Proposition 2.2, then the map ι_m takes the form

$$a \times (a_1, \dots, a_m) \mapsto a \times (1, 0, \dots, 0) \times_{\mathbb{C}^*} (a_1, \dots, a_m)$$

which is indeed the inclusion of the fiber over $[1 : 0 : \cdots : 0]$. \square

Next is a useful corollary to Proposition 2.2.

Corollary 2.4. *For $n = 2$ and m even, $Rat_1(Gr(2, m)) \simeq S^2 \times S^{2m-1}$.*

Proof. Identify \mathbb{C}^2 with \mathbb{H} the quaternions and let it act on $\mathbb{C}^m = (\mathbb{C}^2)^{\frac{m}{2}}$ via quaternionic multiplication on each factor. Since \mathbb{H} is a division algebra, this action restricts to an action of $\mathbb{C}^2 - \{0\}$ onto $\mathbb{C}^m - \{0\}$. If we write $Rat_1(Gr(2, m))$ as $(\mathbb{C}^2 - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^m - \{0\})$ as in Proposition 2.2, then this quotient is diffeomorphic to $\mathbb{P}^1 \times (\mathbb{C}^m - \{0\})$ via the map sending the class $[v, x] \mapsto ([v], v.x)$, with $v.x$ meaning v acting on x via quaternionic multiplication. \square

Corollary 2.4 can also be derived from a general description of $Rat_1(Gr(2, m))$ as an attachment space. Consider the sphere fibration obtained from Lemma 2.3;

$$(2) \quad S^{2m-1} \longrightarrow Rat_1(Gr(2, m)) \longrightarrow S^2, \quad m \geq 2$$

This bundle has a section since the euler class of its associated bundle lies in the trivial group $H^{2m}(S^2)$. We quickly review some classical but structural results on sphere bundles over spheres due to James-Whitehead and Sasao (see [24]). We make use of a bit of notation: (i) by an unlabeled map $S^{k-1} \longrightarrow \Omega S^k$ we mean the map adjoint to the identity of S^k , and (ii) a fibration $E \longrightarrow B$ with fiber F has a holonomy map $h : \Omega B \longrightarrow Aut(F)$; with $Aut(F)$ being the self-homotopy equivalences of F and which classifies the fibration up to fiber homotopy type (Stasheff). Let E be the total space of an S^k -bundle over another sphere S^n . Then the composite

$$(3) \quad f : S^{n-1} \times S^k \longrightarrow \Omega S^n \times S^k \xrightarrow{h} S^k$$

determines the bundle completely up of fiber homotopy equivalence, where h is the holonomy again. When the bundle has a section, this map factors through the half-smash $S_+^{n-1} \wedge S^k \simeq S^k \vee S^{n-1} \wedge S^k \longrightarrow \Omega S_+^n \wedge S^k$ and hence defines a map of the same name

$$(4) \quad f : S^{n-1} \wedge S^k \longrightarrow S^k$$

which in turn defines by adjunction an element $[f] \in \pi_{n-1}(Aut(S^k)) = \pi_{n-1}(\Omega_{\pm 1}^k S^k) \cong \pi_{n+k-1}(S^k)$. If $\iota_k : S^k \hookrightarrow S^k \vee S^n$ is the inclusion of the first factor, we will write $[\iota_k \circ f]$ the homotopy class in $\pi_{n+k-1}(S^k \vee S^n)$.

Proposition 2.5. *If E is an S^k -bundle over S^n with a section, then up to homotopy*

$$E = (S^k \vee S^n) \bigcup_{[\iota_k, \iota_n] + [\iota_k \circ f]} e^{n+k}$$

Proof. Write as in ([25], proposition 2) the total space E in the form

$$E = D^n \times S^k \cup S^k / (x, y) \sim f(x, y), \quad (x, y) \in S^{n-1} \times S^k$$

We can then think of E as the CW complex $S^k \cup_{\alpha} e^n \cup_{\beta} e^{k+n}$ with $\alpha(x) = f(x, *)$ and f as in (3), α' is the characteristic map for e^n ; i.e. $\alpha = \partial\alpha'$, and

$$\beta : S^{k+n-1} = D^n \times S^{k-1} \cup S^{n-1} \times D^k \longrightarrow D^n \times * \cup S^{n-1} \times S^k \xrightarrow{\alpha' \cup f} e^n \cup_{\alpha} S^k$$

If the bundle has a section, then α is null-homotopic and β becomes

$$(5) \quad D^n \times S^{k-1} \cup S^{n-1} \times D^k \longrightarrow D^n \times * \cup S^{n-1} \times S^k \xrightarrow{\phi \times * \cup * \times f} S^n \times * \cup * \times S^k = S^n \vee S^k$$

where ϕ is collapsing the boundary of the disk. As we pointed out and in the presence of a section, the map f factors through $S^{n-1} \wedge S^k = S^{n+k-1} \longrightarrow S^k$. By construction the homotopy class of (5) is the element $[\iota_k, \iota_n] + i_{k*}([f]) \in \pi_{n+k-1}(S^n \vee S^k)$. \square

We use this Proposition to completely determine the homotopy type of $\text{Rat}_1(\text{Gr}(2, m))$ and recover in particular Corollary 2.4.

Corollary 2.6. *$\text{Rat}_1(\text{Gr}(2, m))$ is up to homotopy the CW complex*

$$(S^2 \vee S^{2m-1}) \bigcup_{[\iota_2, \iota_{2m-1}] + m[\iota_{2m-1}] \circ \eta} e^{2m+1}$$

where $\eta \in \pi_{2m}(S^{2m-1}) \cong \mathbb{Z}_2$ is the (Hopf) generator.

Proof. $\text{Rat}_1(\text{Gr}(2, m))$ is up to homotopy the Borel construction $S^3 \times_{S^1} S^{2m-1}$. The projection onto S^2 has a section². The existence of the section identifies $\text{Rat}_1(\text{Gr}(2, m))$ with an adjunction space $(S^2 \vee S^{2m-1}) \bigcup_g e^{2m+1}$ where $g : S^{2m} \xrightarrow{\alpha} S^{2m-1} \hookrightarrow S^2 \vee S^{2m-1}$. According to (4), the map α is adjoint to a map $S^1 \rightarrow \Omega_1^{2m-1} S^{2m-1}$. To understand this map, consider the diagram

$$\begin{array}{ccccccc} & & & U(m) & & & \\ & & \nearrow h_1 & \downarrow & \searrow J_m & & \\ S^1 & \xrightarrow{\iota} & \Omega S^2 & \xrightarrow{h} & L_1^{2m-1} S^{2m-1} & \xrightarrow{\sigma} & \Omega_1^{2m} S^{2m} \\ & & \searrow h_2 & \uparrow & \nearrow & & \\ & & & \Omega_1^{2m-1} S^{2m-1} & & & \end{array}$$

where h is the holonomy of the bundle (2), which factors through h_2 , h_1 is the holonomy of the bundle $m\mathcal{O}(-1)$ of which (2) is the sphere bundle, σ is suspension and finally J_m is the map given by considering $U(m)$ as self-transformations of \mathbb{C}^m then compactifying. The map $\alpha : S^{2m} \longrightarrow S^{2m-1}$ is by construction the adjoint to $h_2 \circ \iota$. We claim that α is up to homotopy $-m\eta$ where η is the representative of the Hopf generator in $\pi_{2m}(S^{2m-1})$. To see this, it is enough to show that the composite $J_m \circ h_1 \circ \iota$ is adjoint to $-m\eta$ as well.

The bundle $m\mathcal{O}(-1)$ over S^2 is classified by a *clutching number* in $\pi_1(U(m)) = \mathbb{Z}$ represented by $h_1 \circ \iota$. This clutching number is $-m \in \mathbb{Z}$ since this clutching map is the composite $S^1 \rightarrow U(1)^m \rightarrow U(m)$ sending

$$\lambda \longmapsto (\lambda^{-1}, \dots, \lambda^{-1}) \longmapsto \begin{pmatrix} \lambda^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{-1} \end{pmatrix}$$

The composite of this map with the determinant map $U(m) \rightarrow S^1$ sends $\lambda \longmapsto \lambda^{-m}$ from which we deduce that the clutching number is $-m$. So what remains to be seen is that J_m sends the generator

²The set of all such sections is in one to one correspondence with S^1 -equivariant maps of S^3 into S^{2m-1} .

of $\pi_1(U(m))$ to the hopf generator in $\pi_{2m+1}(S^{2m})$. But this is a direct consequence of the diagram

$$\begin{array}{ccc} U(1) = S^1 & \xrightarrow{J_1} & \Omega_1^2 S^2 \\ \downarrow \subset & & \downarrow \sigma \\ U(m) & \xrightarrow{J_m} & \Omega_1^{2m} S^{2m} \end{array}$$

and the fact that the adjoint of J_1 can be seen to be a Hopf generator by looking at the linking number. \square

Proposition 2.7. *Write $y, z \in H^*(\text{Rat}_1(\text{Gr}(2, m); \mathbb{Z}_2))$ the generators in degrees $2m - 1$ and $2m + 1$ respectively. Then $Sq^2(y) = mz$, with Sq^2 is the Steenrod squaring operation.*

Proof. This comes down to showing that for m odd, $Sq^2 y = z$. But this is equivalent to having the attaching map of the $2m + 1$ -cell in $\pi_{2m}(S^2 \vee S^{2-1})$ project onto the Hopf class in $\pi_{2m}(S^{2m-1})$. \square

3. THE QUADRIC GRASSMANNIAN

The Grassmann variety $\text{Gr}(2, 2)$ can be realized via the Plucker embedding $\wp : \text{Gr}(2, 2) \hookrightarrow \mathbb{P}^5$ as a hypersurface of degree 2 given in homogeneous coordinates by $z_0 z_1 + z_2 z_3 + z_4 z_5 = 0$. There is a general fact we now recall. If $X \subset \mathbb{P}^n$ is a nonsingular hypersurface of degree d , then its diffeomorphism type is determined completely by n and d . In fact two such hypersurfaces X, Y in \mathbb{P}^n are ambiently isotopic in the sense that there is a diffeomorphism of \mathbb{P}^n , isotopic to the identity, which restricts to give a diffeomorphism of X to Y (see [17], §4). Consequently we can choose the hyperquadric defined by the ‘‘Fermat’’ equation

$$Q_n := \{z_0^2 + \cdots + z_{n+1}^2 = 0\}$$

as our model hypersurface (this is simply connected if $n \geq 1$). Since $\text{Gr}(2, 2)$ is also a hyperquadric in \mathbb{P}^5 , it is diffeomorphic to Q_4 . This ambient diffeomorphism being needed later is made explicit below.

Lemma 3.1. *The selfmap $\gamma : \mathbb{P}^5 \rightarrow \mathbb{P}^5$ which takes $[z_0 : \dots : z_5]$ to*

$$\left[\frac{z_1 - z_0}{2}, i \frac{(z_0 + z_1)}{2}, \frac{z_3 - z_2}{2}, i \frac{(z_2 + z_3)}{2}, \frac{z_5 - z_4}{2}, i \frac{(z_4 + z_5)}{2} \right]$$

restricts to a diffeomorphism between $\text{Gr}(2, 2)$ and Q_4 .

Remark 3.2. An abstract homeomorphism $Q_4 \cong \text{Gr}(2, 2)$ can be obtained as follows. Let $G^+(2, 2n) \cong SO(2n+2)/SO(2) \times SO(2n)$ be the real grassmann variety of oriented 2-planes in \mathbb{R}^{2n+2} . Then there is a diffeomorphism $G^+(2, 2n) \cong Q_{2n}$, for all $n \geq 1$, which sends an oriented 2-plane spanned by orthonormal vectors v_1, v_2 to $\pi(v_1 + iv_2)$ in Q_{2n} , where π is the natural projection $\mathbb{C}^{2n+2} \setminus \{0\} \rightarrow \mathbb{P}^{2n+1}$ (eg. [18]). It remains to identify $\text{Gr}(2, 2)$ with $G^+(2, 4)$ the real Grassmann manifold of all oriented two plane subspaces of \mathbb{R}^6 , and this can be done through the sequence of homeomorphisms:

$$\begin{aligned} \text{Gr}(2, 2) := SU(4)/S(U(2) \times U(2)) &\cong Spin(6)/(U(1) \times SU(2) \times SU(2)) \\ &\cong Spin(6)/(SO(2) \times Spin(4)) \\ &\cong SO(6)/(SO(2) \times SO(4)) =: G^+(2, 4) \end{aligned}$$

using the standard group isomorphisms $U(1) \cong SO(2)$, $SU(4) \cong Spin(6)$ and $Spin(4) \cong SU(2) \times SU(2)$. Notice the cute result that Q_4 , and hence $\text{Gr}(2, 2)$, has the same homology and same homotopy groups as $\mathbb{P}^2 \times S^4$ according to [18].

In all cases and as a consequence of the above we can derive the following result.

Proposition 3.3. *There is a non-multiplicative splitting*

$$\Omega \text{Gr}(2, 2) \simeq S^1 \times S^3 \times \Omega S^5 \times \Omega S^7$$

Proof. By non-multiplicative we mean that there is no H -map from the left to the righthand side which is a homotopy equivalence. The H -space structure on the righthand side is the obvious product of H -space structures. The proof of this Theorem is based on an observation from [22]. Replace $Gr(2, 2)$ by the Fermat hypersurface $Q_4 = \{\sum z_i^2 = 0\}$ and consider the pullback of the Hopf fibration

$$(6) \quad \begin{array}{ccc} \tilde{Q}_4 & \longrightarrow & S^{11} \\ \downarrow & & \downarrow \\ Q_4 & \hookrightarrow & \mathbb{P}^5 \end{array}$$

Write $u_i = \text{Re}(z_i)$ and $v_i = \text{Im}(z_i)$. Then

$$S^{11} = \left\{ (u_1, v_1, \dots, u_6, v_6) \mid \sum u_j^2 + \sum v_j^2 = 2 \right\}$$

The pullback is given by

$$\tilde{Q}_4 = \left\{ (u_1, v_1, \dots, u_6, v_6) \mid \sum u_j^2 = \sum v_j^2 = 1 \mid \sum u_j v_j = 0 \right\}$$

and this is evidently homeomorphic to the space of orthonormal 2-frames in R^6 , or equivalently to the unit tangent bundle $ST(S^5)$ to S^5 . This is an S^1 -bundle over $Gr(2, 2)$ and so we therefore have the fibration

$$(7) \quad \Omega ST(S^5) \longrightarrow \Omega Gr(2, 2) \xrightarrow{p} S^1$$

Note that the projection p induces an isomorphism on fundamental groups as can easily be checked. The fibration (7) has a homotopy retract $S^1 \rightarrow \Omega Gr(2, 2)$. A justification of this fact goes as follows: suppose X is a simply connected space with $\pi_2(X) = \mathbb{Z}$. Represent the adjoint of the generator in $\pi_2(X)$ by a map $\alpha : S^1 \rightarrow \Omega X$. Since $\pi_1(\Omega X) \cong \mathbb{Z}$, then $H^1(\Omega X) \cong \mathbb{Z}$ and this is represented by a map $p : \Omega X \rightarrow S^1$ which is necessarily a homotopy retract to α ⁽³⁾.

In all cases, the existence of a homotopy section for the multiplicative fibration (7) means that $\Omega Gr(2, 2) \simeq S^1 \times \Omega ST(S^5)$. On the other hand, $ST(S^5) \rightarrow S^5$ also has a section and thus

$$(8) \quad \begin{aligned} \Omega Gr(2, 2) \simeq S^1 \times \Omega ST(S^5) &\simeq S^1 \times \Omega S^4 \times \Omega S^5 \\ &\simeq S^1 \times S^3 \times \Omega S^7 \times \Omega S^5 \end{aligned}$$

Here we use as well the fact that $\Omega S^4 \simeq S^3 \times \Omega S^7$.

To show that our splitting is not multiplicative, we look at the loop homology algebra of $Gr(2, 2)$ and show that it differs from the loop homology algebra of the righthand side of the decomposition. To that end one uses the fact that if X is a simply connected space of finite type that is *formal* over \mathbb{Z} , then its cohomology algebra determines completely its Pontryagin loop algebra structure. To explain what this means, a space is formal if its cohomology algebra is quasi-isomorphic to its singular cochain algebra. This means that there are quasi-isomorphisms of differential graded (associative) algebras

$$(C^*(X), d) \leftarrow (A, d) \rightarrow (H^*(X), 0)$$

There is a very useful criterion given by [10] for showing when a simply connected finite type space X is formal when working with coefficients over a field or over \mathbb{Z} if X has torsion free homology. This consists in showing that the minimal Adams-Hilton model for X has a purely quadratic differential. We recall that the Adams-Hilton model is a differential graded algebra of the form TV , a tensor algebra on $V = s^{-1}\tilde{H}_*(X)$ (coefficients in a commutative ring and s is the suspension operator) with differential d which decomposes as $d_2 + d_3 + \dots$, where d_k is the part that maps into length- k decomposables. We say that TV has a purely quadratic differential if $d_k = 0$, $k > 2$.

On the other hand for any simply connected X , we have the quasi-isomorphism of algebras

$$C_*(\Omega X) \simeq \underline{\Omega}(C_*(X)) \simeq (BC^*(X))^\vee$$

³This also follows from the following fact in homotopy theory: Let $\Omega E \xrightarrow{p} \Omega B \xrightarrow{p} F \xrightarrow{i} E \xrightarrow{f} B$ be the sequence of fibrations induced from f . Then i is homotopically trivial if and only if p has a homotopy section.

where $\underline{\Omega}$ is Adams cobar construction, B is the bar construction, ${}^\vee$ the hom-dual and $C_*(X)$ are the singular chains. The Pontryagin ring structure only depends on the algebra structure on $C^*(X)$, so that if X is formal we can replace $C^*(X)$ by $H^*(X)$ as algebras. In other words, for formal spaces the ring structure of $H_*(\Omega X)$ depends only on the cohomology ring $H^*(X)$. In particular if a formal space X has the cohomology algebra of a product of spheres, then its loop homology is the loop homology of that product of spheres.

In the case of $ST(S^5)$, it is clear that for its minimal model, $V = T(e_3, e_4, e_8)$ and $d(e_8)$ is necessarily purely quadratic (in fact $d(e_8) = [e_3, e_4]$). It is then clear that $ST(S^5)$ is formal. On the other hand, an easy argument using the Leray-Hirsch Theorem shows that $H^*(ST(S^5)) \cong H^*(S^4 \times S^5)$ as algebras. This implies as we discussed that

$$H_*(\Omega ST(S^5)) \cong H_*(\Omega S^4) \otimes H_*(\Omega S^5)$$

as Pontryagin rings. But $H_*(\Omega S^4) \cong T(a_3)$ is obviously not isomorphic as algebras to $H_*(S^3 \times \Omega S^7)$, because there is an exterior 3-dimensional generator only for the latter, so that the decomposition in (8) cannot be one of H -spaces. \square

Remark 3.4. There is a bundle $U(2) \longrightarrow V(2, 2) \longrightarrow Gr(2, 2)$, where $V(2, 2)$ is the Stiefel manifold of orthonormal 2-frames in \mathbb{C}^4 . One can recover the decomposition (8) from this bundle and the known homotopy equivalence $V(2, 2) \simeq S^5 \times S^7$. Note that the argument of proof above shows that the projection $\Omega Gr(2, 2) \longrightarrow U(2)$ cannot have a multiplicative section [1].

Remark 3.5. Many questions remain open. We suspect strongly that $\Omega ST(S^5) \simeq \Omega S^4 \times \Omega S^5$ and $\Omega Gr(2, 2) \simeq S^1 \times \Omega S^4 \times \Omega S^5$ as H -spaces. Interestingly the induced splitting

$$\Omega_0^2 Gr(2, 2) \simeq \Omega S^3 \times \Omega^2 S^5 \times \Omega^2 S^7$$

is *not* a splitting of double loop spaces [1]. To our knowledge, the structure of $H_*(\Omega^2 Gr(2, 2); \mathbb{Z}_p)$ as a module over the Dyer-Lashof algebra is not generally known.

3.1. The Homology Embedding ι_* . Using the pullback bundle (6), we can try to take another view on the geometry of the inclusion

$$\iota : \text{Rat}_1(Gr(2, 2)) \hookrightarrow \Omega_1^2(Gr(2, 2))$$

According to Corollary 2.4, $\text{Rat}_1(Gr(2, 2)) \simeq S^2 \times S^3$ while

$$(9) \quad \Omega_0^2 Gr(2, 2) \simeq \Omega^2 ST(S^5) \simeq \Omega^2 S^4 \times \Omega^2 S^5 \simeq \Omega S^3 \times \Omega S^7 \times \Omega^2 S^5$$

Write α, β the homology classes of degree 2, 3 respectively in $H_*(\text{Rat}_1(Gr(2, 2))) \cong H_*(S^2 \times S^3)$, and we let a, b and f the bottom homology generators of degree 2, 3 and 5 in $H_*(\Omega_0^2 Gr(2, 2))$ according to the decomposition (9). View $S^5 \subset \mathbb{R}^6$ and let $s : S^5 \longrightarrow ST(S^5)$ be the section sending locally

$$v = (v_1, \dots, v_6) \mapsto (v, v^\perp), \quad v^\perp = (v_2, -v_1, v_4, -v_3, v_6, -v_5)$$

Proposition 3.6. *The following diagram commutes*

$$\begin{array}{ccccccc} S^3 & \xrightarrow{\simeq} & \text{Rat}_1(\mathbb{P}^2) & \xrightarrow{\iota_2} & \text{Rat}_1(Gr(2, 2)) & & \\ \downarrow & & \downarrow & & \downarrow \iota & & \\ \Omega^2 S^5 & \xrightarrow{\simeq} & \Omega_1^2(\mathbb{P}^2) & \xrightarrow{\iota_2} & \Omega_1^2(Gr(2, 2)) & \longrightarrow & \Omega^2 ST(S^5) \end{array}$$

and the point is that the bottom composite is homotopic to the section $\Omega^2 s$. As a consequence $i_*(\beta) = b$ in homology.

Proof. The claim immediately follows from Lemma 2.3 and the existence of the commutative diagram

$$\begin{array}{ccccc}
 S^1 & \xrightarrow{=} & S^1 & \xrightarrow{=} & S^1 \\
 \downarrow & & \downarrow & & \downarrow \\
 S^5 & \xrightarrow{s} & ST(S^5) & \longrightarrow & S^{11} \\
 \downarrow & & \downarrow \pi & & \downarrow \\
 \mathbb{P}^2 & \xrightarrow{\beta} & Q_4 & \longrightarrow & \mathbb{P}^5 \\
 & \searrow & \downarrow \cong & & \uparrow \gamma' \\
 & & Gr(2, 2) & \xrightarrow{\gamma} & \mathbb{P}^5
 \end{array}$$

where the top part of the diagram is made of circle fibrations, where γ' is a self-diffeomorphism, $ST(S^5)$ is the sphere tangent bundle identified with the pullback of the Hopf fibration over \mathbb{P}^5 , Q_4 is the Fermat hypersurface ambiently diffeomorphic to $Gr(2, 2)$ via γ (Lemma 3.1), and $\beta : [w_1, w_2, w_3] \mapsto [\bar{w}_1 : i\bar{w}_1 : \bar{w}_2 : i\bar{w}_2 : \bar{w}_3, i\bar{w}_3]$. The expression of β comes from the fact found in the proof of Proposition 3.3 that if $v = (v_1, \dots, v_6) \in S^5$, then

$$\pi(v, v^\perp) = [v_1 - iv_2 : v_2 + iv_1 : v_3 - iv_4 : v_4 + iv_3 : v_5 - iv_6 : v_6 + iv_5]$$

An inspection shows that all maps in the upper part of the diagram commute. Remains to see the bottom part of the diagram and what γ' is. The composite $\mathbb{P}^2 \hookrightarrow Gr(2, 2) \xrightarrow{\gamma} \mathbb{P}^5$ is the standard embedding $[w_1 : w_2 : w_3] \mapsto [w_1 : 0 : w_2 : 0 : w_3 : 0]$. If we apply γ to this we obtain

$$[w_1 : w_2 : w_3] \mapsto [iw_1 : w_1 : iw_2 : w_2 : iw_3 : w_3]$$

This differs by an obvious self-diffeomorphism α of \mathbb{P}^5 from the top map β explicited above. We can then set $\gamma' = \gamma \circ \alpha$. \square

We can also trace the effect of the map ι on H_2 . If we write $\text{Rat}_1(Gr(2, 2)) \simeq S^3 \times_{S^1} S^3$, then as in the proof of Lemma 2.4 there is an equivalence $S^3 \times_{S^1} S^3 \longrightarrow S^2 \times S^3$ sending $[a, b] \mapsto ([a], ab)$. There is then a section $S^2 \longrightarrow S^3 \times_{S^1} S^3$, $[a] \mapsto [a, a^{-1}]$ where $a \in S^3$ and $[a]$ its class in $S^2 = \mathbb{P}^1$. Consider the bundle $U(2) \longrightarrow V(2, 2) \longrightarrow Gr(2, 2)$ with holonomy $h : \Omega Gr(2, 2) \longrightarrow U(2)$. Since $U(2) \simeq S^1 \times SU(2)$, we write $\Omega_c U(2) \simeq \Omega SU(2)$ a component of $\Omega U(2)$. The following is a consequence of a calculation in ([19], equation 4.8).

Proposition 3.7. *The composite*

$$\mathbb{P}^1 \xrightarrow{s} \text{Rat}_1(Gr(2, 2)) \xrightarrow{\iota} \Omega_1^2(Gr(2, 2)) \xrightarrow{\Omega h} \Omega SU(2) = \Omega S^3$$

is up to homotopy the adjoint to the identity of S^3 . In homology, this means that $\iota_(\alpha) = a$.*

4. PROOF OF THEOREM 1.1

The tautological bundle γ_n over $Gr(n, m)$ is the n -dimensional complex vector bundle with total space

$$\{(P, v) \in Gr(n, m) \times \mathbb{C}^{n+m} \mid P \in Gr(n, m), v \in P\}$$

Define $\mathcal{O}(d)$ to be the complex line bundle over \mathbb{P}^n whose total space is the borel construction

$$\mathbb{C} \times_{\mathbb{C}^*} (\mathbb{C}^{n+1} - \{0\}) := \mathbb{C} \times (\mathbb{C}^{n+1} - \{0\}) / \sim$$

which is the quotient obtained by identifying tuples

$$(x, z_0, \dots, z_n) \sim (\lambda^d x, \lambda z_0, \dots, \lambda z_n) \quad \lambda \in \mathbb{C}^*, d \in \mathbb{Z}$$

Clearly the trivial line bundle is $\epsilon = \mathcal{O}(0)$. It can be checked that

- $\text{Hom}_{\mathbb{C}}(\mathcal{O}(n), \epsilon) = \mathcal{O}(-n)$, and that

- $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2) \cong \mathcal{O}(d_1 + d_2)$, more particularly $\mathcal{O}(\pm d) := \mathcal{O}(\pm 1)^{\otimes d}$.
- The tautological bundle γ_1 over \mathbb{P}^n is isomorphic to $\mathcal{O}(-1)$. This bundle has *no global* non-zero sections. The bundle γ_1 over $\mathbb{R}P^1$ the real Grassmann manifold $Gr(1, 1)$ is homeomorphic to the Mobius band. The bundle γ_1 is called the *Hopf* line bundle since the complement of its zero section is the bundle γ_1^* with total space

$$\mathbb{C}^* \times_{\mathbb{C}^*} (\mathbb{C}^{n+1} - \{0\}) = \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$$

and this is precisely the Hopf fibration.

- The dual of the Hopf line bundle is the *hyperplane* bundle H . This is the bundle obtained by projecting $\mathbb{P}^{n+1} - \{[0 : \dots : 0 : 1]\} \longrightarrow \mathbb{P}^n$ sending $[z_0 : \dots : z_n : z] \mapsto [z_0 : \dots : z_n]$. It can easily be seen that $H \cong \mathcal{O}(1)$ and is the dual to the Hopf line bundle.

Definition 4.1. If η is a bundle over \mathbb{P}^n , we write $n\eta$ its n -fold Whitney sum and write η^* the complement of its zero section.

Proof. (of Theorem 1.1) The linear action of $U(n+m)$ on \mathbb{C}^{n+m} induces an action on $Gr(n, m)$ and hence an action on $\text{Hol}_k(Gr(n, m))$ by postcomposition. As in §2, if we choose a decomposition $\mathbb{C}^{n+m} \cong U \oplus Y$ where $U \cong \mathbb{C}^n$ and $Y \cong \mathbb{C}^m$, then the stabilizer of this decomposition is a copy of $U(n) \times U(m)$ embedded in standard diagonal way in $U(n+m)$. The action of $U(n+m)$ on $\text{Hol}(Gr(n, m))$ being transitive, we have a Borel type description

$$\text{Hol}(Gr(n, m)) = \text{Rat}(Gr(n, m)) \times_{U(n) \times U(m)} U(n+m)$$

If we write a map $f \in \text{Rat}_k(Gr(n, m))$ as a rational function (1), then the action of $X \times Y \in U(n) \times U(m)$ on the $A_i \in \text{Hom}(U, Y)$ is given by the product of matrices

$$(X, Y) \cdot A_i = Y^{-1} A_i X$$

If we identify $\text{Rat}_1(Gr(n, m))$ with $\mathbb{C} \times (\mathbb{C}^m - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^n - \{0\})$ as in Proposition 2.2, the action of $U(n) \times U(m)$ translates into a diagonal action on this product by sending

$$(10) \quad (X, Y) \times (a, \mathbf{v}, \mathbf{w}) \longmapsto (a, (Y^{-1} \mathbf{v}, X^T \mathbf{w}))$$

with $X \in U(n), Y \in U(m), \mathbf{v} \in \mathbb{C}^m, \mathbf{w} \in \mathbb{C}^n$.

We can then rewrite up to homotopy

$$(11) \quad \begin{aligned} \text{Hol}_1(Gr(n, m)) &= \text{Rat}_1(Gr(n, m)) \times_{U(n) \times U(m)} U(n+m) \\ &\simeq [(\mathbb{C}^n - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^m - \{0\})] \times_{U(n) \times U(m)} U(n+m) \end{aligned}$$

where both the actions of $U(n)$ and $U(m)$ on $(\mathbb{C}^n - \{0\})$ and $(\mathbb{C}^m - \{0\})$ are via multiplication on the right. This is not quite the same action as in (10) but it yields homeomorphic quotients. Define

$$\begin{aligned} X(n, m) &:= [(\mathbb{C}^m \times_{\mathbb{C}^*} (\mathbb{C}^n - \{0\})) \times_{U(m) \times U(n)} U(n+m)] \\ &= \mathbb{C}^m \times_{\mathbb{C}^* \times U(m)} [(\mathbb{C}^n - \{0\}) \times_{U(n)} U(n+m)] \\ &\simeq \mathbb{C}^m \times_{U(1) \times U(m)} \left[\frac{U(n)}{U(n-1)} \times_{U(n)} U(n+m) \right] \\ &= \mathbb{C}^m \times_{U(1) \times U(m)} \left[\frac{U(n+m)}{U(n-1)} \right] \end{aligned}$$

where here we have replaced $\mathbb{C}^n - \{0\}$ up to equivariant homotopy by $S^{2n-1} = U(n)/U(n-1)$ and \mathbb{C}^* by $U(1) = S^1$. The projection

$$(12) \quad X(n, m) \longrightarrow U(m+n)/U(1) \times U(n-1) \times U(m)$$

makes $X(n, m)$ into an m -dimensional complex vector bundle over $Fl_{(1,n)}(\mathbb{C}^{n+m})$ of which (11) is the sphere bundle. Our aim is therefore to show that (12) is a bundle isomorphic to $p_1^*(H) \otimes p_2^*(Q)$.

The action of $\lambda \in U(1)$ on $S^{2n-1} \subset \mathbb{C}^n$ is via multiplication by λ . This means that the action of the same λ on \mathbb{C}^m is via multiplication by λ^{-1} . That is, the action of $(\lambda, A) \in U(1) \times U(m)$ on $\vec{w} \in \mathbb{C}^m$ is

$\lambda^{-1}A(\mathbf{w})$, and the action of $U(1)$ on $U(n+m)/U(n-1)$ is via left multiplication if we view $U(1)$ as the top left standard matrix subgroup of $U(n+m)$.

Note that multiplication $U(1) \times U(n-1) \longrightarrow U(n)$ induces the projection $p_2 : Fl_{(1,n)}(\mathbb{C}^{n+m}) \rightarrow Gr(n, m)$ and the total space of the pullback of Q is precisely

$$(13) \quad \mathbb{C}^m \times_{U(m)} [U(n+m)/U(1) \times U(n-1)]$$

with $U(m)$ acting via multiplication on the left. Similarly the total space of the pullback of the dual hyperplane bundle via the projection p_1 induced from $U(n-1) \times U(m) \rightarrow U(n+m-1)$ is

$$(14) \quad \mathbb{C} \times_{U(1)} [U(n+m)/U(n-1) \times U(m)]$$

with $U(1)$ acting by multiplication by the inverse. We rewrite both bundles (13) and (14) as in

$$\begin{aligned} & \mathbb{C}^m \times_{U(1) \times U(m)} [U(n+m)/U(n-1)] \\ & \mathbb{C} \times_{U(1) \times U(m)} [U(n+m)/U(n-1)] \end{aligned}$$

So that their tensor product is the bundle

$$(\mathbb{C} \otimes \mathbb{C}^m) \times_{U(1) \times U(m)} [U(n+m)/U(n-1)]$$

with $(\lambda, A) \in U(1) \times U(m)$ acting on $z \otimes \mathbf{w}$ by $\lambda^{-1}z \otimes A\mathbf{w}$. Upon identifying $\mathbb{C} \otimes \mathbb{C}^m$ with \mathbb{C}^m , we recover precisely the bundle $X_{n,m}$ and the theorem is proved.

The proof of Theorem 1.1 shows that for $n \leq m$ there is a diagram of fibrations

$$\begin{array}{ccccc} S^{2m-1} & \xrightarrow{=} & S^{2m-1} & & \\ \downarrow & & \downarrow & & \\ Rat_1(Gr(n, m)) & \longrightarrow & Hol_1(Gr(n, m)) & \xrightarrow{ev} & Gr(n, m) \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{P}^{n-1} & \longrightarrow & Fl(1, n, n+m) & \longrightarrow & Gr(n, m) \end{array}$$

where the left vertical fibration is same as the sphere bundle of $m\mathcal{O}(-1)$ over \mathbb{P}^{n-1} . \square

We can now derive a few interesting corollaries.

Corollary 4.2. *$Hol_1(Gr(n, m))$ is of the homotopy type of a closed oriented manifold of dimension $2m - 1 + \dim Fl_{(1,2)}(\mathbb{C}^{n+m}) = 2n(m+1) + 2m - 3$.*

Corollary 4.3. [15] *$Hol_1(\mathbb{P}^m)$ is up to homotopy the unit tangent bundle of \mathbb{P}^m .*

Proof. In this case $Fl_{(1,1)}(\mathbb{C}^{1+m}) = \mathbb{P}^m$ and Hol_1 is up to homotopy the total space of the sphere bundle of $Q^\vee \otimes \gamma$. Here Q^\vee is the notation for the dual of the anti-tautological bundle Q . We need check that this tensor product is isomorphic to the tangent bundle of \mathbb{P}^n . We know that $Q \oplus \gamma \cong (n+1)\epsilon$ the trivial bundle of rank $(n+1)$ over \mathbb{P}^n and hence by taking duals $Q^\vee \oplus \gamma^\vee \cong (n+1)\epsilon$. We can tensor both sides by γ and get

$$(Q^\vee \otimes \gamma) \oplus \epsilon = (n+1)\gamma$$

using the fact that $\gamma^\vee \otimes \gamma = \epsilon$. On the other hand, it is quite well-known that

$$T\mathbb{P}^n \oplus \epsilon \cong (n+1)\gamma$$

This means that $T\mathbb{P}^n$ and $Q^\vee \otimes \gamma$ are stably isomorphic. But within this range the two bundles must be isomorphic according to [13], chapter 9, Theorem 1.5. \square

The next corollary recovers a calculation of Cohen-Lupercio-Segal [5] who identify $Rat_1(BU(n))$ with $F_{1,n} \simeq \mathbb{P}^{n-1}$ the first piece in the Mitchell filtration of $\Omega SU(n)$. Here $Rat_1(BU(n))$ is defined as the direct limit of the inclusions $Rat_1(Gr(n, m)) \hookrightarrow Rat_1(Gr(n, m+1))$.

Corollary 4.4. *There are homotopy equivalences $\text{Hol}_1(BU(n)) \simeq Fl_{(1,n)}(\mathbb{C}^\infty)$ and $\text{Rat}_1(BU(n)) \simeq \mathbb{P}^{n-1}$. Moreover $H^*(\text{Hol}_1(BU(n)))$ is a free $H^*(BU(n))$ -module with generators $1, \zeta, \dots, \zeta^{n-1}$.*

Proof. Theorem 1.1 gives a fibration

$$S^{2m-1} \longrightarrow \text{Hol}_1(Gr(n, m)) \longrightarrow Fl_{(1,n)}(\mathbb{C}^{n+m})$$

which we can stabilize via the embeddings $Fl_{(1,n)}(\mathbb{C}^{n+m}) \longrightarrow Fl_{(1,n)}(\mathbb{C}^{n+m+1})$ that take a $(1, n)$ -flag in \mathbb{C}^{n+m} and embed it in \mathbb{C}^{n+m+1} . In the limit we get the diagram of fibrations

$$\begin{array}{ccccc} S^\infty & \longrightarrow & \text{Rat}_1(BU(n)) & \longrightarrow & \mathbb{P}^{n-1} \\ \downarrow = & & \downarrow & & \downarrow \\ S^\infty & \longrightarrow & \text{Hol}_1(BU(n)) & \longrightarrow & Fl_{(1,n)}(\mathbb{C}^\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & BU(n) & \xrightarrow{\cong} & Gr(n, \infty) \end{array}$$

and the first two claims follow at once since S^∞ is contractible. The cohomological calculation is on the other hand a general consequence of the Leray-Hirsh theorem (see for example Switzer's book, Theorem 15.47). If $\mathbb{P}^k \xrightarrow{j} E \longrightarrow B$ is a fibration with an element $\zeta \in H^2(E)$ such that $j^*(\zeta)$ maps to the generator of $H^2(\mathbb{P}^k)$, and thus j^* is an epimorphism, then $H^*(E)$ is a free $H^*(B)$ -module with generators $1, \zeta, \dots, \zeta^k$. \square

Remark 4.5. The Mitchell filtration consists of compact complex subvarieties

$$F_{1,n} \subset F_{2,n} \subset \dots \subset F_{k,n} \subset \dots \subset F_{\infty,n} = \Omega_{pol} SU(n) \simeq \Omega SU(n)$$

where $F_{k,n}$ is homeomorphic to the space of polynomial loops $\gamma : S^1 \longrightarrow SU(n)$ such that $\gamma(1) = 1$ and $\gamma(z) = \sum_0^k A_i z^i$ where A_i are $n \times n$ matrices. In [23] it is stated that $F_{1,n}$ corresponds to the subspace generated by all transformations $\lambda_V : z \longrightarrow \begin{pmatrix} z & 0 \\ 1 & 0 \end{pmatrix}$ with matrix written in terms of the decomposition $\mathbb{C}^n = V \oplus V^\perp$, with $\dim V = 1$. This is a copy of \mathbb{P}^{n-1} . Note that the inclusion $F_{1,n} \hookrightarrow \Omega_{pol} SU(n)$ is up to homotopy the standard map $\mathbb{P}^{n-1} \longrightarrow \Omega SU(n)$ which generates in homology the ring $H_*(\Omega SU(n))$.

5. THE UNPARAMETERIZED LINEAR MAPS

In this section we prove Theorem 1.2. We defined in the introduction the spaces $\widetilde{\text{Hol}}_1(Gr(n, m))$ and $\widetilde{\text{Rat}}_1(Gr(n, m))$ of unparameterized maps. Recall that we can associate to a morphism $f : \mathbb{P}^1 \rightarrow Gr(n, m)$ its *kernel* $\ker(f) = \bigcap_{p \in \mathbb{P}^1} f(p)$. Similarly the *span* of f is the linear span of these subspaces. Naturally $\ker(f) \subset \text{span}(f) \subset \mathbb{C}^{n+m}$. Notice that both *ker* and *span* are invariant under the action of $PGL_2(\mathbb{C}) \simeq PU(2)$ on $\text{Hol}_1(Gr(n, m))$.

Consider the based case first. Recall any such map is of the form $z \mapsto f(z) = \frac{A}{z-a}$ with $a \in \mathbb{C}$ and A is an $n \times m$ -matrix of rank one which we write $A = \mathbf{v} \cdot \mathbf{w}^T$ for some non-zero $\mathbf{w} \in \mathbb{C}^m$ and $\mathbf{v} \in \mathbb{C}^n$. As in Proposition 2.2, we have the identification

$$(15) \quad \text{Rat}_1(Gr(n, m)) \cong \mathbb{C} \times (\mathbb{C}^n - \{0\}) \times_{\mathbb{C}^*} (\mathbb{C}^m - \{0\})$$

Write $(\alpha, \beta) \in \mathbb{C}^* \times \mathbb{C}$ the element of $Aff(\mathbb{C})$ acting on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ by fixing ∞ and sending $z \mapsto \alpha z + \beta$. The action of $Aff(\mathbb{C})$ on $\text{Rat}_1(Gr(n, m))$ by precomposition of maps translates under the above identifications to the action

$$(\alpha, \beta) \times (a, [\mathbf{v}, \mathbf{w}]) \longmapsto \left(\frac{a - \beta}{\alpha}, \left[\frac{1}{\alpha} \mathbf{v}, \mathbf{w} \right] \right)$$

where again (α, β) is viewed as an element of $Aff(\mathbb{C})$ and $(a, [\mathbf{v}, \mathbf{w}])$ as a rational map according to (15). The quotient by this action is the quotient of $(\mathbb{C}^n - \{0\}) \times (\mathbb{C}^m - \{0\})$ by $\mathbb{C}^* \times \mathbb{C}^*$ acting as follows:

$(\alpha, a) \times (\mathbf{v}, \mathbf{w}) \mapsto (\frac{1}{a\alpha}\mathbf{v}, a\mathbf{w})$. This is equivalent to the action by multiplication componentwise so that the quotient is $\widetilde{\text{Rat}}_1(\text{Gr}(n, m)) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ and the projection $\text{Rat}_1(\text{Gr}(n, m)) \longrightarrow \widetilde{\text{Rat}}_1(\text{Gr}(n, m))$ sends $(a, \mathbf{v}, \mathbf{w}) \mapsto ([\mathbf{v}], [\mathbf{w}])$. We need to see next how to relate this map to the kernel and span. Our first claim is that for $f \in \text{Rat}_1(\text{Gr}(n, m))$, $\dim \ker(f) = n - 1$ and $\dim \text{span}(f) = n + 1$. The basing is $f(\infty) = E_n \subset \mathbb{C}^{n+m}$, where by definition E_n is the plane spanned by the *first* n -coordinates vectors. We then have $\ker(f) \subset E_n \subset \text{span}(f)$. The subspace of $Fl_{(n-1, n+1)}(\mathbb{C}^{n+m})$ denoted Fl_{E_n} of all $(A \subset B)$ flags such that $A \subset E_n \subset B$ is easily identified with $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ and we have the map

$$(16) \quad \widetilde{\text{Rat}}_1(\text{Gr}(n, m)) \longrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \quad , \quad f \longmapsto (\ker(f) \subset \text{span}(f))$$

which we claim must now be a homeomorphism. We check the details next.

First we compute the dimension of kernel and span. To that end we need know that the map $f : z \mapsto \frac{1}{z-a}A$ as described in (1), with $A = [v_1, \dots, v_n] \cdot [w_1, \dots, w_m]^T$ of rank one, can also be viewed as the map sending z to the n -plane $f(z)$ in $\text{Gr}(n, m)$ given as the span of the row vectors in the matrix

$$(17) \quad \begin{pmatrix} 1 & 0 & \cdots & 0 & v_1 w_1 & \cdots & v_1 w_m \\ 0 & 1 & \cdots & 0 & v_2 w_1 & \cdots & v_2 w_m \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z-a & v_n w_1 & \cdots & v_n w_m \end{pmatrix}$$

This matrix is well-defined up to the left action by $GL_n(\mathbb{C}[z])$ which corresponds to taking row operations. By dividing up by $z-a$ and letting $z \rightarrow +\infty$, we see that indeed $f(\infty) = E_n \subset \mathbb{C}^{n+m}$. We recall from Remark 2.1 that $T(z) = \frac{1}{z-a}A$ is the transfer function associated to (17). Out of this matrix representation (17), $\text{span}(f)$ becomes by definition the span of the row vectors making up (17) as z varies. This is easily seen to be the $(n+1)$ -dimensional subspace spanned by the $(n+1)$ -row vectors

$$\begin{Bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_1 & \cdots & w_m \end{Bmatrix}$$

and is uniquely determined by the homogeneous vector $[w_1 : \cdots : w_m]$. On the other hand the kernel can be seen to correspond to

$$\ker(f) = \left\{ \alpha_1 \begin{pmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \sum \alpha_i v_i = 0 \right\}$$

and this is an $(n-1)$ -dimensional subspace determined by the homogeneous vector $[v_1 : \cdots : v_n]$. The pair $(\ker(f), \text{span}(f))$ such that $\ker(f) \subset E_n \subset \text{span}(f)$ determines a unique point in $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ and this sets up an explicit homeomorphism with $\widetilde{\text{Rat}}_1(\text{Gr}(n, m))$.

We now turn to the unbased holomorphic maps. Note that $U(n+m)$ acts on all of $\text{Hol}_1(\text{Gr}(n, m))$ and $\widetilde{\text{Hol}}_1(\text{Gr}(n, m))$ by acting on the target and that the (\ker, span) -map is $U(n+m)$ -equivariant. We will construct directly an inverse map ϕ to this (\ker, span) map

$$(18) \quad \phi : Fl_{(n-1, n+1)}(\mathbb{C}^{n+m}) \longrightarrow \widetilde{\text{Hol}}_1(\text{Gr}(n, m))$$

and hence show that ϕ is a homeomorphism. Such a map exists when restricted to the subspace $Fl_{E_n} \cong \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ of flags $(A \subset E_n \subset B)$ as we've just shown and we write ϕ_r such a restriction. Let $(E_{n-1} \subset E_{n+1})$ be the trivial flag, where the basis vectors for E_k are taken to be the first k -coordinate

vectors. Then $\phi_r(E_{n-1} \subset E_{n+1})$ is the class under the $Aff(\mathbb{C})$ -action of the map $z \mapsto \frac{1}{z} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$ corresponding to picking up $[w_1 : \cdots : w_m] = [1 : \cdots : 0 : 0]$ and $[v_1 : \cdots : v_n] = [0 : \cdots : 0 : 1]$. Equivalently this is the map which in matrix form is

$$(19) \quad z \mapsto \text{span row vectors} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & z & 1 & \cdots & 0 \end{pmatrix}$$

the righthand side being an $n \times (n+m)$ -matrix.

A unitary matrix $g \in U(n+m)$ acts on a flag $(A \subset B) \in Fl_{(n-1, n+1)}(\mathbb{C}^{n+m})$ by sending it to the flag $(g(A) \subset g(B))$. Given any such flag $(A \subset B)$, there is an element g of $U(n+m)$ sending it to $(E_{n-1} \subset E_{n+1})$. We can then construct ϕ in (18) by setting

$$(20) \quad \phi(A \subset B) = g^{-1} \phi_r(g(A \subset B)) = g^{-1} \phi_r(E_{n-1} \subset E_{n+1})$$

We have to prove this map is well-defined. Any other element $h \in U(n+m)$ taking $(A \subset B)$ to the trivial flag satisfies the property that $hg^{-1} \in U(m-1) \times U(2) \times U(n-1)$ as the subgroup of $U(n+m)$ consisting of blocks $\begin{pmatrix} U(n-1) & & \\ & U(2) & \\ & & U(m-1) \end{pmatrix}$. Different choices of matrices in $U(n-1)$ and $U(m-1)$ do not affect $\phi(A \subset B)$ since they stabilize the flag $E_{n-1} \subset E_n \subset E_{n+1}$ and hence don't affect $\phi_r(E_n \subset E_{n+1})$. It remains to analyze the effect of $U(2)$ on (20). Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$. This stabilizes the trivial flag but takes E_{n+1} from its standard representation as span of the first $n+1$ -coordinate vectors to the span

$$E'_{n+1} = \text{span row vectors} \begin{pmatrix} (Id_{n-1}) & & & & \\ & a & b & 0 & \cdots & 0 \\ & c & d & 0 & \cdots & 0 \end{pmatrix}$$

Of course $E'_{n+1} = E_{n+1}$ but we distinguish them in notation since a priori the map ϕ can depend on this choice of basis. According to (20), $(E_n \subset E'_{n+1})$ must be mapped to the holomorphic map obtained by acting on each row vector of (19) by $g \in U(2) \subset U(n+m)$. This gives the map

$$z \mapsto \text{span row vectors} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & az+b & cz+d & \cdots & 0 \end{pmatrix}$$

or equivalently the map

$$z \mapsto \text{span row vectors} \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{az+b}{cz+d} & 1 & \cdots & 0 \end{pmatrix}$$

This is now evidently the composition of the map in (19) with the transformation $z \mapsto \frac{az+b}{cz+d}$. This means in summary that the candidate $\phi(A \subset B)$ we constructed in (20) is well-defined as an element of $\widetilde{\text{Hol}}_1(Gr(n, m))$ and is clearly an inverse to the (\ker, span) -map.

Remains to show this map is continuous but this is a consequence of the fact that ϕ is the induced quotient map in the following diagram

$$\begin{array}{ccc} Fl_{E_n} \times U(n+m) & & \\ \downarrow \pi & \searrow \tilde{\phi} & \\ Fl_{(n-1, n+1)}(\mathbb{C}^{n+m}) & \xrightarrow{\phi} & \widetilde{Hol}_1(Gr(n, m)) \end{array}$$

where $\tilde{\phi}(f, g) = g\phi_r(f)$ and $\pi(f, g) = gf$. Theorem 1.2 is proved.

6. HOMOLOGICAL CALCULATIONS

This final section uses the sphere bundle description in Theorem 1.1 to calculate the homology of $Hol_1(Gr(n, m))$ for various ring coefficients and small values of n and m . We will be working with an explicit description for $H^*(Gr(n, m); \mathbb{Z})$. This cohomology is generated by the Chern classes of the tautological bundle of $Gr(n, m)$. Consider the two natural embeddings $Gr(n, m) \hookrightarrow BU(n)$ and $Gr(n, m) \hookrightarrow BU(m)$. The cohomology of $Gr(n, m)$ is generated by the pullbacks of the chern classes $c_i \in H^*(BU(n))$ and $\bar{c}_j \in H^*(BU(m))$. These pullback classes satisfy the relation

$$(21) \quad (1 + c_1 + \dots + c_n)(1 + \bar{c}_1 + \dots + \bar{c}_m) = 1$$

This relation is the consequence of the fact that the total chern classes multiply as in $c(\gamma_m)c(\gamma_n) = c(\gamma_m \oplus \gamma_n)$ and that $\gamma_m \oplus \gamma_n = \gamma_m \oplus Q_m \cong \epsilon_{n+m}$ is trivial on $Gr(n, m)$. The following is standard.

Proposition 6.1. *There is a graded ring isomorphism*

$$H^*(Gr(n, m)) = \mathbb{Z}[c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_m] / \left(\sum_{i+j=k} c_i \cdot \bar{c}_j; 1 \leq k \leq n+m \right)$$

with $\deg c_i = \deg \bar{c}_j = 2i$.

Remark 6.2. The relations $\sum_{i=0}^k c_i \times \bar{c}_{k-i} = 0$ for $k = 1, \dots, m$ can be solved inductively to express $\bar{c}_1, \dots, \bar{c}_m$ in terms of c_1, \dots, c_n , and the ring $H^*(Gr(n, m))$ is in fact a quotient of $\mathbb{Z}[c_1, \dots, c_n]$ by an explicit ideal (ρ_1, \dots, ρ_m) . That is and for $m \geq n \geq 2$, $H^*(Gr(n, m)) = \mathbb{Z}[c_1, \dots, c_n] / (\rho_1, \dots, \rho_m)$ with $\deg \rho_i = 2m + 2i$, $1 \leq i \leq n$, and the ρ_i making up a so-called *regular sequence* (complete formulae in [4], §2). In particular and when $n = 2$, $H^*(Gr(2, m)) = \mathbb{Z}[c_1, c_2] / (\rho_1, \dots, \rho_m)$ with

$$(22) \quad \rho_i = \sum_{p_1+2p_2=m+i} (-1)^{p_1+p_2} \binom{p_1+p_2}{p_1} c_1^{p_1} c_2^{p_2}$$

We work out explicitly the case of the quadric Grassmann manifold $G(2, 2)$.

Corollary 6.3. $H^*(Gr(2, 2)) = \mathbb{Z}[c_1, c_2] / (c_1^3 - 2c_1c_2, c_1^4 - 2c_2^2)$.

Proof. We “peel off” the relations $(1 + c_1 + c_2)(1 + \bar{c}_1 + \bar{c}_2) = 1$ one by one. In homological degree 2, $c_1 + \bar{c}_1 = 0$ so that $\bar{c}_1 = -c_1$. In degree four, $c_1\bar{c}_1 + c_2 + \bar{c}_2 = 0$ so that $\bar{c}_2 = c_2 - c_1^2$. In degree six $c_1\bar{c}_2 + \bar{c}_1c_2 = 0$ leading to the first relation $c_1^3 - 2c_1c_2 = 0$. The relation $c_1^4 = 2c_2^2$ follows similarly. This is of course consistent with (22). \square

More generally we have the following result of Baum needed in §6.1 when computing the cohomology of some flag manifolds.

Theorem 6.4. *If G is a compact connected Lie group and U a closed connected subgroup, both of whose integral cohomology rings are exterior algebras on (odd degree) generators, then the sequence*

$$H^*(BG) \xrightarrow{\rho^*} H^*(BU) \xrightarrow{\sigma^*} H^*(G/U)$$

has the property that the kernel of σ^ is the ideal of $H^*(BU)$ generated by the elements of positive degree in $\text{Image}(\rho^*)$.*

Remark 6.5. $H^*(Gr(n, m))$ being a quotient of $H^*(BU(n))$ shows that $H_*(Gr(n, m))$ injects into $H_*(BU(n))$, and thus the embedding $Gr(n, m) \hookrightarrow Gr(n, m+1)$ induces a monomorphism in homology. This fact is no longer true for real Grassmann varieties.

6.1. The Quadric Grassmannian. In this section we give complete rational calculations for $Hol_1(Gr(2, 2))$ where $Gr(2, 2)$ is the quadric grassmannian. We do this by first understanding the Gysin sequence for the bundle ξ derived from Theorem 1.1

$$(23) \quad S^3 \longrightarrow Hol_1(G(2, 2)) \longrightarrow Fl(1, 2, 4)$$

This is given by (setting $Fl := Fl(1, 2, 4)$ and $G = Gr(2, 2)$)

$$(24) \quad \dots \rightarrow H^{i+3} Hol_1 G \longrightarrow H^i FL \xrightarrow{\chi} H^{i+4} Fl \longrightarrow H^{i+4} Hol_1 G \rightarrow \dots$$

where χ is the cup-product with the Euler class of $Fl(1, 2, 4)$; that is with the top chern class of the bundle $p_1^*(\gamma) \otimes p_2^*(Q^\vee)$ in Theorem 1.1.

To begin we need the cohomology structure of $Fl(1, 2, 4) = U(4)/U(1) \times U(1) \times U(2)$. Notice that there is a fibration

$$(25) \quad \mathbb{P}^1 \longrightarrow Fl(1, 2, 4) \longrightarrow G(2, 2)$$

which is trivial in cohomology (additively) since all generators are concentrated in even degree. This says that $H^*(Fl(1, 2, 4))$ is torsion free with Poincaré polynomial

$$(26) \quad \begin{aligned} \mathcal{P}(Fl(1, 2, 4)) &= \mathcal{P}(\mathbb{P}^1) \cdot \mathcal{P}(G(2, 2)) \\ &= (1 + t^2)(1 + t^2 + 2t^4 + t^6 + t^8) \\ &= 1 + 2t^2 + 3t^4 + 3t^6 + 2t^8 + t^{10} \end{aligned}$$

To get the cohomology structure however we have to resort to the result of Baum explained in Theorem 6.4.

Lemma 6.6. *There is a ring isomorphism*

$$\begin{aligned} H^*(Fl(1, 2, 4)) &= \mathbb{Z}[x, y]/((x+y)(x^2+y^2), x^3y+xy^3+x^2y^2) \\ &= \mathbb{Z}[x, y]/((x+y)(x^2+y^2), x^4, y^4) \end{aligned}$$

where $|x| = 2, |y| = 2$.

Proof. Consider the short exact sequence

$$H^*(BU(4)) \xrightarrow{\rho^*} H^*(BU(1) \times BU(1) \times BU(2)) \xrightarrow{\sigma^*} H^*(Fl(1, 2, 4))$$

with kernel of σ^* the ideal of positive degree elements in ρ^* . Here $H^*(BU(1) \times BU(1) \times BU(2)) = \mathbb{Z}[x, y, z_1, z_2]$ so that

$$H^*(Fl(1, 2, 4)) = \mathbb{Z}[x, y, z_1, z_2]/I$$

where I is generated by $(1+x)(1+y)(1+z_1+z_2)$; i.e with the ideal of relations generated by

$$1 + (x+y+z_1) + (z_2+xy+xz_1+yz_1) + (yz_2+xz_2+xyz_1) + xyz_2 = 1$$

From this we can deduce that $z_1 = -x-y$, $z_2 = -xz_1-xy-yz_1 = x^2+y^2+xy$ and that $(x+y)(x^2+y^2) = 0$ et $x^3y+xy^3+x^2y^2 = 0$ as claimed. The second isomorphism follows by identifying generators in each dimension as in (26). \square

Remark 6.7. As we have indicated there are two maps

$$p_1 : Fl(1, 2, 4) \longrightarrow \mathbb{P}^3 \quad \text{and} \quad p_2 : Fl(1, 2, 4) \longrightarrow Gr(2, 2)$$

and so we determine their effect on cohomology. We write $H^*(Gr(2, 2)) = \mathbb{Z}[c_1, c_2]/I$ and $H^*(\mathbb{P}^3) = \mathbb{Z}[u]/(u^4)$. The first map is modeled after

$$U(4)/U(1) \times U(1) \times U(2) \longrightarrow U(4)/U(1) \times U(3)$$

which is induced from the multiplication $U(1) \times U(2) \longrightarrow U(3)$. There is a diagram

$$\begin{array}{ccc} Fl(1, 2, 4) & \longrightarrow & BU(1) \times BU(1) \times BU(2) \\ \downarrow & & \downarrow \pi \\ \mathbb{P}^3 & \longrightarrow & BU(1) \end{array}$$

where π is projection onto the first component. Since the generator u is induced from that single copy of $BU(1)$, $f^*(u) = x$. Similarly p_2 is determined from multiplication $H^*(BU(2)) \longrightarrow H^*(BU(1) \times BU(1))$ which sends $1 + c_1 + c_2$ to $(1 + x)(1 + y)$ so that

$$p_2^*(c_1) = x + y \quad , \quad p_2^*(c_2) = xy$$

We are now in a position to read off the Euler class of the bundle (23), or equivalently the top chern class c_2 of $\xi := p_1^*(\gamma) \otimes p_2^*(Q^\vee)$.

Lemma 6.8. $c_2(\xi) = 3x^2 + y^2 + 2xy \in H^4(Fl(1, 2, 4)) = \mathbb{Z}\{x^2, xy, y^2\}$.

Proof. By the formula for the total chern class of a tensor product of two bundles L, E with L a line bundle, one gets the general relation $c_2(E \otimes L) = c_2(E) + c_1(E) \times c_1(L) + c_1^2(L)$ (see [8], p.55). Applying this to our situation with $L = p_1^*(\gamma)$ and $E = p_2^*(Q^\vee)$, we get

$$c_2(\xi) = (x^2 + y^2 + xy) + x(x + y) + x^2 = 3x^2 + y^2 + 2xy$$

as claimed. \square

An analysis of the Gysin sequence (24) with integral coefficients yields the following complete calculation stated in Theorem 1.3.

Theorem 6.9. $\tilde{H}^*(Hol_1(Gr(2, 2)); \mathbb{Z})$ is given by

i	2	4	6	7	8	9	11	13
H^i	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}_4	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	\mathbb{Z}

and 0 in all other degrees.

Proof. In the Gysin sequence (24), the map χ is multiplication by $3x^2 + y^2 + 2xy$. Since the cohomology of $Fl(1, 2, 4)$ is concentrated in even dimension, the exact sequence splits into short exact sequences

$$0 \rightarrow H^{2i+1}Hol_1G(2, 2) \rightarrow H^{2i-2}Fl(1, 2, 4) \xrightarrow{\chi} H^{2i+2}Fl(1, 2, 4) \rightarrow H^{2i+2}Hol_1G(2, 2) \rightarrow 0$$

from which we deduce that

$$\begin{aligned} H^{odd}Hol_1(G(2, 2)) &= \ker(H^{odd-3}Fl(1, 2, 4) \longrightarrow H^{odd+1}Fl(1, 2, 4)) \\ H^{even}Hol_1(G(2, 2)) &= \operatorname{coker}(H^{even-4}Fl(1, 2, 4) \longrightarrow H^{even}Fl(1, 2, 4)) \end{aligned}$$

Now we list generators for $H^*(Fl(1, 2, 4))$

$$\begin{aligned} H^2 &= \mathbb{Z}\{x, y\} \\ H^4 &= \mathbb{Z}\{x^2, xy, y^2\} \\ H^6 &= \mathbb{Z}\{x^3, x^2y, y^2x\} \quad y^3 = -x^3 - x^2y - xy^2 \\ H^8 &= \mathbb{Z}\{x^3y, y^3x\} \quad x^4 = y^4 = 0 \quad , \quad x^2y^2 = -xy^3 - x^3y \\ H^{10} &= \mathbb{Z}\{x^3y^2\} \quad x^3y^2 = -x^2y^3 \end{aligned} \tag{27}$$

where here $H^* = H^*(Fl(1, 2, 4))$ and we've written the relations on the right for each degree, and then analyze multiplication by the euler class on each. It is clear that $H^1Hol_1G(2, 2) = 0$ and $H^2Hol_1G(2, 2) = H^2Fl(1, 2, 4) = \mathbb{Z}\{x, y\}$. To compute H^5 and H^6 for example we consider the portion of the sequence

$$0 \rightarrow H^5Hol_1G(2, 2) \rightarrow H^2Fl(1, 2, 4) \xrightarrow{\chi} H^6Fl(1, 2, 4) \rightarrow H^6Hol_1G(2, 2) \rightarrow 0$$

with $H^2 Fl(1, 2, 4) = \mathbb{Z}\{x, y\}$ and $H^6 Fl(1, 2, 4) = \mathbb{Z}\{x^3, x^2y, xy^2\}$. In the bases (x, y) and (x^3, x^2y, xy^2) , the matrix of χ is $\begin{pmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ with Smith associated form $\begin{pmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$. Thus the cokernel is $\mathbb{Z} \oplus \mathbb{Z}_4$ and $H^5 Hol_1 G(2, 2) = 0$. Similarly

$$0 \rightarrow H^7 Hol_1 G(2, 2) \rightarrow H^4 Fl(1, 2, 4) \xrightarrow{\chi} H^8 Fl(1, 2, 4) \rightarrow H^8 Fl(1, 2, 4) \rightarrow 0$$

with $H^4 Fl(1, 2, 4) = \mathbb{Z}^3\{x^2, xy, y^2\}$, $H^8 Fl(1, 2, 4) = \mathbb{Z}^2\{x^3y, xy^3\}$ and $x^2y^2 = -x^3y - xy^3$. Keeping in mind the relation $x^2y^2 = -x^3y - xy^3$ we see that

$$\begin{aligned} \chi(x^2) &= 3x^4 + x^2y^2 + 2x^3y = -x^3y - xy^3 + 2x^3y = x^3y - xy^3 \\ \chi(xy) &= 3x^3y + xy^3 + 2x^2y^2 = 3x^3y + xy^3 + 2(-x^3y - xy^3) \\ &= x^3y - xy^3 \\ \chi(y^2) &= 3x^2y^2 + y^4 + 2xy^3 = 3(-x^3y - xy^3) + 2xy^3 = -3x^3y - xy^3 \end{aligned}$$

The matrix of χ is $\begin{pmatrix} 1 & 1 & -3 \\ -1 & -1 & -1 \end{pmatrix}$ with associated Smith form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$ and hence $H^7 Hol_1 G(2, 2) = \mathbb{Z}$ et $H^8 Hol_1 G(2, 2) = \mathbb{Z}_4 \oplus \mathbb{Z}$. The rest follows similarly. \square

Corollary 6.10. *The rational Poincaré series for the holomorphic mapping space is*

$$\mathcal{P}(Hol_1 G(2, 2)) = 1 + 2t^2 + 2t^4 + t^6 + t^7 + 2t^9 + 2t^{11} + t^{13}$$

Remark 6.11. From (23) (or corollary 4.2), we know that $Hol_1(Gr(2, 2))$ is of the homotopy type of a closed manifold of real dimension 13. This explains the Poincaré duality for the coefficients appearing in Corollary 6.10.

6.2. The general case $(1, 2, n+2)$ -flags. We try to extend the results on the quadric grassmannian to maps into $Gr(2, n)$. We generalize lemma 6.6 first

Lemma 6.12. *We have the algebra isomorphism*

$$H^* Fl(1, 2, n+2) = \mathbb{Z}[x, y] / \left(\sum_{i+j=n+1} x^i y^j, x^{n+2} = 0 = y^{n+2} \right)$$

Proof. As in the proof of Lemma 6.6, $H^* Fl(1, 2, n+2) = \mathbb{Z}[x, y]/I$ where I is the ideal generated by all homogeneous relations in x and y derived from

$$(1+x)(1+y)(1+z_1+z_2+\dots+z_n) = 1$$

These we get recursively

$$(xy) z_{k-2} + (x+y) z_k + z_k = 0, \quad \forall k > 1$$

We can write this down in matrix form

$$\begin{pmatrix} z_k \\ z_{k-1} \end{pmatrix} = \begin{pmatrix} -(x+y) & -xy \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_{k-1} \\ z_{k-2} \end{pmatrix} = \begin{pmatrix} -(x+y) & -xy \\ 1 & 0 \end{pmatrix}^{k-1} \cdot \begin{pmatrix} -(x+y) \\ 1 \end{pmatrix}$$

After diagonalizing we obtain the power matrix

$$\begin{pmatrix} -(x+y) & -xy \\ 1 & 0 \end{pmatrix}^k = \frac{1}{y-x} \begin{pmatrix} (-x)^{k+1} - (-y)^{k+1} & y(-x)^{k+1} - x(-y)^{k+1} \\ (-x)^k - (-y)^k & y(-x)^k - x(-y)^k \end{pmatrix}$$

from which we deduce all relations

$$\begin{aligned} z_n &= (-1)^n \sum_{i+j=n} x^i y^j \\ z_{n+1} &= 0 = -(xy) z_{n-1} - (x+y) z_n \implies \sum_{i+j=n+1} x^i y^j = 0 \end{aligned}$$

$$z_{n+2} = 0 = -(xy) z_n - (x+y) z_{n+1} \implies xy \sum_{i+j=n} x^i y^j = 0$$

so that

$$H^* Fl(1, 2, n+2) = \mathbb{Z}[x, y] / \left(\sum_{i+j=n+1} x^i y^j, xy \sum_{i+j=n} x^i y^j \right)$$

Now we need see that both ideals

$$\left(\sum_{i+j=n+1} x^i y^j, xy \sum_{i+j=n} x^i y^j \right) \leftrightarrow \left(\sum_{i+j=n+1} x^i y^j, x^{n+2}, y^{n+2} \right)$$

are equivalent but this is immediate. \square

Next we compute the euler class $\mathbf{e}(\xi_n)$ of the sphere fibration (Theorem 1.1)

$$\xi_n : S^{2n-1} \longrightarrow Hol_1(G(2, n)) \longrightarrow Fl(1, 2, n+2)$$

Proposition 6.13. $\mathbf{e}(\xi_n) = \frac{\partial}{\partial x} \sum_{i+j=n+1} x^i y^j$.

Proof. This euler class corresponds to the top chern class $c_n(p_1^* H \otimes p_2^* Q^\vee)$ of $\zeta(\xi_n)$ and this can be computed by the formula

$$\mathbf{e}(\xi_n) = \sum_{i=0}^n c_1^i(p_1^* H) \cdot c_{n-i}(p_2^* Q^\vee)$$

We have to determine the chern classes of the pullback of Q^\vee . A quick inspection of the diagram

$$\begin{array}{ccc} Fl(1, 2, n+2) & \xrightarrow{p_2} & Gr(2, n) \\ \downarrow & & \downarrow \\ BU(1) \times BU(1) \times BU(n) & \longrightarrow & BU(2) \times BU(n) \end{array}$$

shows that the map in cohomology $H^*(Fl(1, 2, n+2)) \longleftarrow H^*(Gr(2, n)) = \mathbb{Z}[c_1, c_2]/I$, with I the ideal generated by the ρ_1, \dots, ρ_n given in (22), sends

$$\begin{aligned} c_1 &\longmapsto x+y \\ c_2 &\longmapsto xy \\ \bar{c}_k = c_k(Q) &\longmapsto z_n = (-1)^k \sum_{i+j=k} x^i y^j \end{aligned}$$

using the fact that the multiplication map $BU(1) \times BU(1) \longrightarrow BU(2)$ in cohomology sends $c_1 \mapsto x+y$ and $c_2 \mapsto xy$. But $c_k(Q^\vee) = (-1)^k \bar{c}_k(Q)$ so that

$$p_2^*(c_k(Q^\vee)) = (-1)^{2k} \sum_{i+j=k} x^i y^j = \sum_{i+j=k} x^i y^j \in H^*(Fl(1, 2, n+2))$$

To show that $\mathbf{e}(\xi_n) = \frac{\partial}{\partial x} \sum_{i+j=n+1} x^i y^j$, we proceed by induction. The case $n=2$ has been verified in lemma 6.8 where we checked that $\mathbf{e}(\xi_2) = 3x^2 + 2xy + y^2$. The proof now proceeds inductively

$$\begin{aligned} \mathbf{e}(\xi_{n+1}) &= c_{n+1}(Q^\vee) + \mathbf{e}(\xi_n) = \sum_{i+j=n+1} x^i y^j + \frac{\partial}{\partial x} \sum_{i+j=n+1} x^i y^j \\ &= \sum_{i+j=n+1} x^i y^j + \sum_{i+j=n+1} i x^i y^j \\ &= \frac{\partial}{\partial x} \sum_{i+j=n+2} x^i y^j \end{aligned}$$

which is what we wanted to prove. \square

As an application we can give the explicit ranks for $H^*(Fl(1, 2, n + 2); \mathbb{Q})$ as well as the euler class of the sphere bundle ξ . It is then possible by use of the Gysin sequence again to obtain complete calculations for $H^*(Hol_1(Gr(2, n)))$ with all field coefficients. This was done for the quadric grassmannian $Gr(2, 2)$ in Corollary 6.10.

Corollary 6.14. . *The rational Betti numbers for $Hol_1(Gr(2, 3))$ are listed below*

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b_i	0	2	0	3	0	3	0	2	1	1	2	0	3	0	3	0	2	0	1

As expected these betti numbers satisfy Poincaré duality.

REFERENCES

- [1] W. Ben Hammouda, Thesis, University of Lille I, November 2011.
- [2] P. Baum, *Cohomology of homogeneous spaces*, Bull. Amer. Math. Soc. **69** (1963), 531–533.
- [3] A.S. Buch, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235.
- [4] N. Chair, *Explicit computations of the intersection numbers on Grassmannians, and on the space of holomorphic maps from \mathbb{P}^1 into $Gr(n, m)$* , J. Geom. Phys. **38** (2001), no. 2, 170–182.
- [5] R. Cohen, E. Lupercio, G. Segal, *Holomorphic spheres in loop groups and Bott periodicity*, Asian J. Math. **3** (1999), no. 4, 801–818.
- [6] F. De Mari, H. Glüsing-Lüeren, *The spaces of improper rational matrices and ARMA-systems of fixed McMillan degree*, J. Math. Systems Estim. Control **5** (1995), 1–19.
- [7] R. J. Duffin, D. Hazony *The Degree of a Rational Matrix Function*, J. Soc. Industrial Applied Math, **11**, no. 3 (1963), 645–658.
- [8] W. Fulton, *Intersection theory*, Ergebnisse der Math. und ihrer Grenzgebiete, 3rd Series, Springer (1998).
- [9] M.A. Guest, A.N. Pressley, *Holomorphic curves in loop groups*, Commun. Math. Phys. **118** (1988), 511–527.
- [10] M. Haouari, *p-Formalité des espaces*, J. Pure and Applied Algebra **78** (1992), 27–47.
- [11] J. Havlicek, *On spaces of holomorphic maps from two copies of the riemann sphere to complex grassmannians*, Thesis Stanford 1992.
- [12] D. Hurtubise, *Spaces of holomorphic maps from \mathbb{P}^1 to complex grassmann manifolds*, Topics in low-dimensional topology (University Park, PA, 1996), World Sci. Publ. (1999), 99–111.
- [13] D. Husemoller, *Fiber bundles*, graduate texts in mathematics, third edition, Springer verlag.
- [14] C. Martin, R. Hermann, *Applications of algebraic geometry to systems theory : the McMillan degree and kronecker indices of transfer functions as topological and holomorphic system invariants*, SIAM J. Control and Optimization **16**, 5, (1978), 743–755.
- [15] S. Kallel, P. Salvatore, *Rational maps and string topology*, Geometry and Topology **10** (2006), 2001–2028.
- [16] F. Kirwan, *On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles*. Ark. Mat., **24** (1986), 221–275.
- [17] R.S. Kulkarni, J.W. Wood, *Topology of nonsingular complex hypersurfaces*, Advances in Math. **35** (1980), 239–263.
- [18] Hon-Fei Lai, *On the topology of the even-dimensional complex quadrics*, Proceedings of AMS **46** (1974), 419–425.
- [19] B. Mann, R.J. Milgram, *Some spaces of holomorphic maps to complex Grassmann manifolds*, J. Diff. Geometry **33** (1991), 301–324.
- [20] B. Mann, R.J. Milgram, *The topology of rational maps to Grassmannians and a homotopy theoretic proof of the kirwan stability theorem*, Contemp. Math. **146** (1993), 251–275.
- [21] N. Perrin, *Courbes rationnelles sur les variétés homogènes*, Ann. Inst. Fourier (Grenoble) **52** (2002), 105–132.
- [22] K.H. Paranjape, V. Srinivas, *Continuous self-maps of quadric hypersurfaces*, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), Hindustan Delhi (1993) 135–148.
- [23] G. Segal, *Loop groups and harmonic maps*, Advances in homotopy theory, London Math. Soc. Lecture Note Ser. **139** (1989), 153–164.
- [24] S. Sasao, *Homotopy types of spherical fibre spaces over spheres*, Pacific J. Math. **52** (1974), 207–219.
- [25] J. Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246.

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